Analisis frekuensi sinyal waktu diskrit

- Deret Fourier untuk sinyal waktu diskrit periodik
- Transformasi Fourier untuk sinyal diskrit aperiodik

Deret Fourier untuk sinyal diskrit periodik

$$x(n+N) = x(n) \qquad N = \text{perioda dasar}$$

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} = \sum_{k=0}^{N-1} c_k s_k$$

$$s_k = e^{j\omega_k n} \qquad \omega_k = \frac{2\pi k}{N} \qquad -\pi \le \omega_k \le \pi$$

$$f_k = \frac{k}{N} \qquad -\frac{1}{2} \le f_k \le \frac{1}{2}$$

$$c(k) = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \qquad c_{k+N} = c_k$$

Contoh Soal 7.3

Tentukan spektrum dari sinyal-sinyal di bawah ini.

a). $x(n) = \cos \dfrac{\pi n}{3}$ b). $\{1, 1, 0, 0\}$ $N = 4$

$\uparrow$

Jawab :

a). $$x(n) = \cos \frac{\pi n}{3} = \cos 2\pi \frac{1}{6} n$$

$$f_o = \frac{1}{6} \quad \rightarrow \quad N = 6$$

$$c(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} = \sum_{n=0}^{5} x(n) e^{-j2\pi kn/6}$$

$$x(n) = \cos 2\pi \frac{1}{6} n = \frac{1}{2} e^{j2\pi n/6} + \frac{1}{2} e^{-j2\pi n/6}$$

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/6}$$

$$c_1 = \frac{1}{2} \qquad c_{-1} = \frac{1}{2} \qquad c_o = c_2 = c_3 = c_4 = 0$$

$$c_5 = c_{-1+6} = c_{-1} = \frac{1}{2}$$

$$c_1 = \frac{1}{2} \qquad c_{-1} = \frac{1}{2} \qquad c_o = c_2 = c_3 = c_4 = 0$$

$$c_5 = c_{-1+6} = c_{-1} = \frac{1}{2}$$

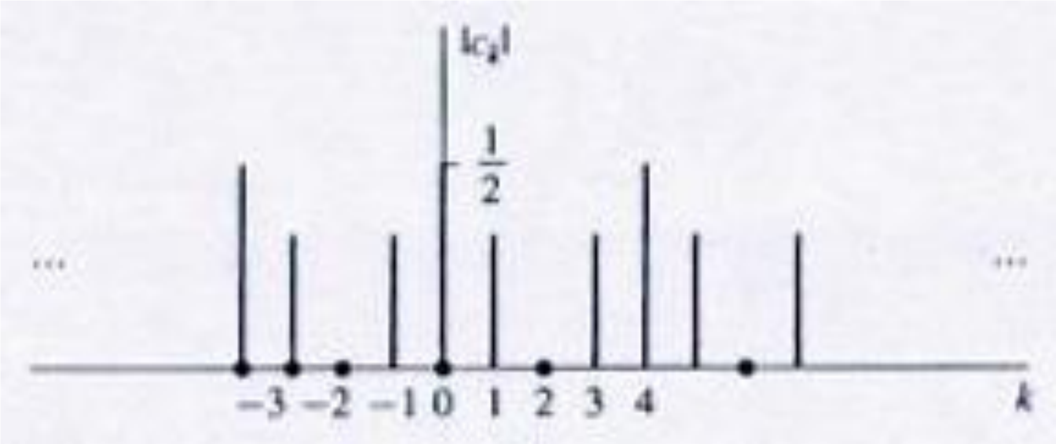
b). $\{\underset{\uparrow}{1}, 1, 0, 0\}$ $N = 4$

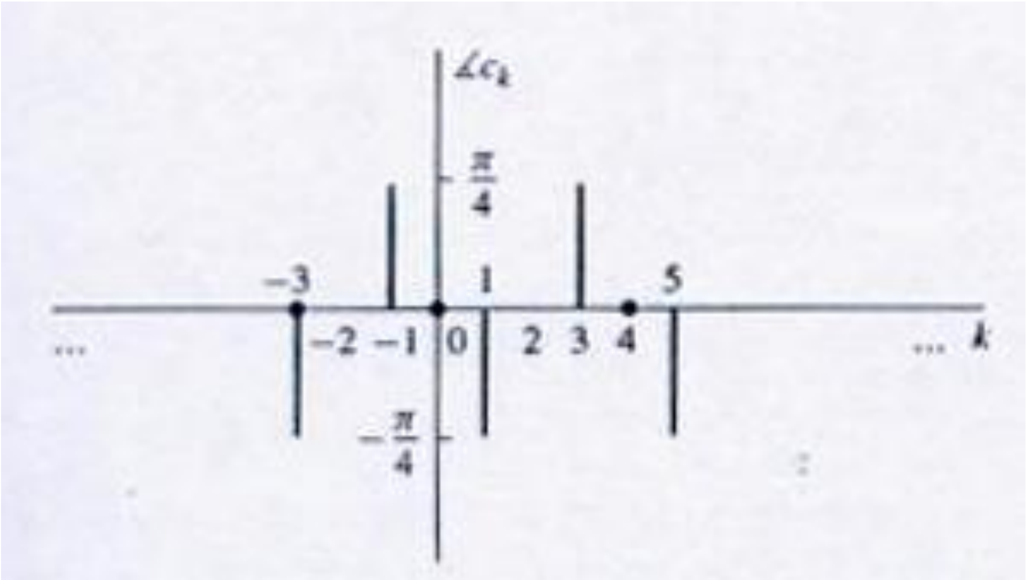
$$c(k) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$$

$$c(k) = \frac{1}{4}\sum_{n=0}^{3} x(n)e^{-j2\pi kn/4} = \frac{1}{4}\left(1 + e^{-j\pi k/2}\right)$$

$$c_o = \frac{1}{2} \quad c_1 = \frac{1}{4}(1 - j) \quad c_2 = 0 \quad c_3 = \frac{1}{4}(1 + j)$$

$$c_o = \frac{1}{2} \qquad c_1 = \frac{1}{4}(1 - j) \qquad c_2 = 0 \qquad c_3 = \frac{1}{4}(1 + j)$$





Contoh Soal 7.4

Tentukan spektrum dari sinyal di bawah ini.

$$x(n) = \cos\frac{2\pi}{3}n + \sin\frac{2\pi}{5}n$$

Jawab :

$$x(n) = \cos\frac{2\pi}{3}n + \sin\frac{2\pi}{5}n = \cos 2\pi\frac{5}{15}n + \sin 2\pi\frac{3}{15}n$$

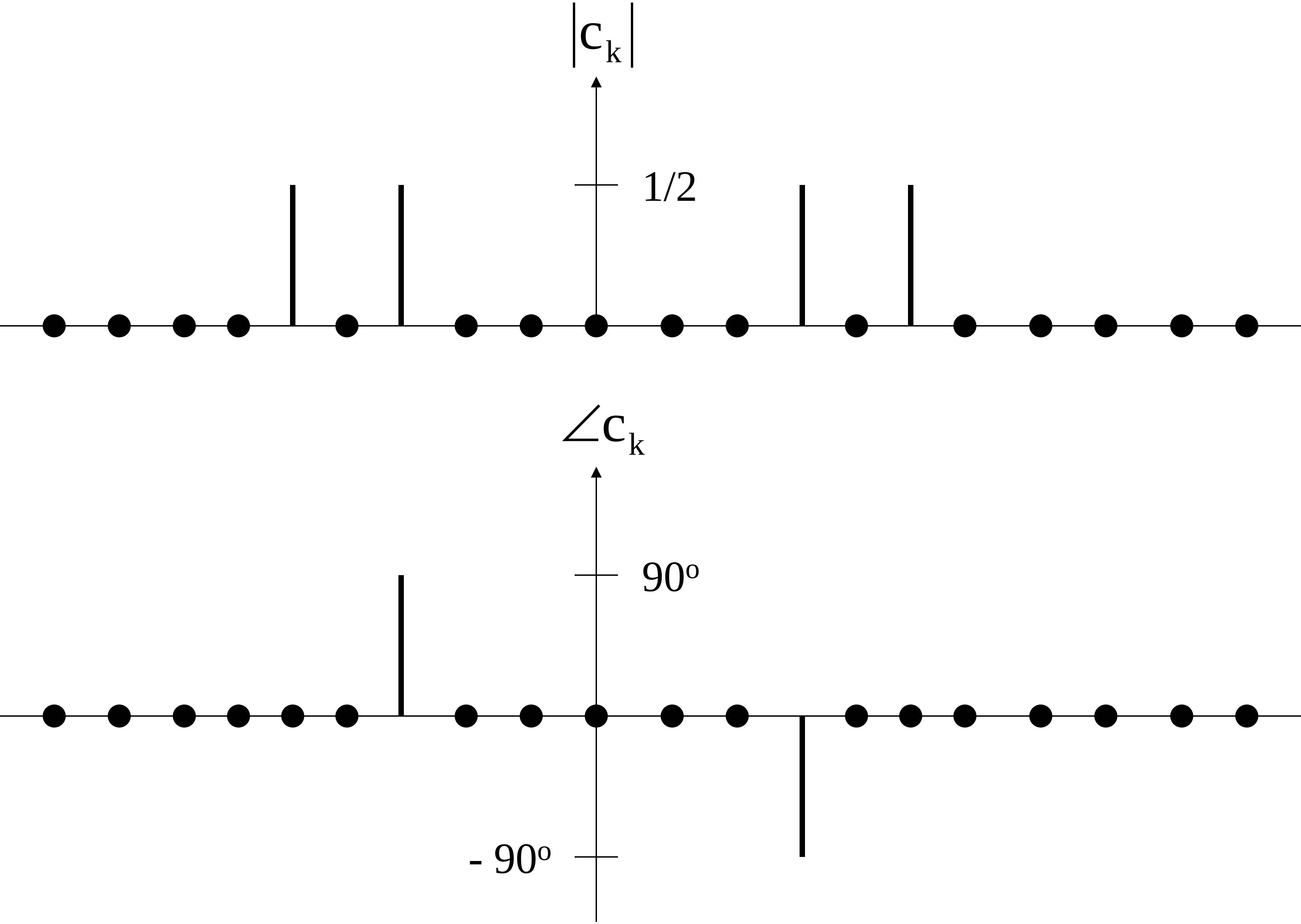
$$x(n) = \frac{e^{j2\pi(5/15)n} + e^{-j2\pi(5/15)n}}{2} + \frac{e^{j2\pi(3/15)n} - e^{-j2\pi(3/15)n}}{2j}$$

$$x(n) = -\frac{j}{2}e^{j2\pi(3/15)n} + \frac{j}{2}e^{-j2\pi(3/15)n} + \frac{1}{2}e^{j2\pi(5/15)n} + \frac{1}{2}e^{-j2\pi(5/15)n}$$

$$x(n) = -\frac{j}{2}e^{j2\pi(3/15)n} + \frac{j}{2}e^{-j2\pi(3/15)n} + \frac{1}{2}e^{j2\pi(5/15)n} + \frac{1}{2}e^{-j2\pi(5/15)n}$$

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} = \sum_{k=0}^{14} c_k e^{j2\pi kn/15}$$

$$c_{-5} = \frac{1}{2} \qquad c_{-3} = \frac{j}{2} \qquad c_3 = -\frac{j}{2} \qquad c_5 = \frac{1}{2}$$



|c_k|
1/2
∠c_k
90°
- 90°

Transformasi Fourier dari sinyal diskrit aperiodik

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$ → **Bentuk Deret Fourier**

$$X(\omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega+2\pi k)n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}e^{-j2\pi kn} = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = X(\omega)$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$

Contoh Soal 7.6

Tentukan sinyal diskrit yang transformasi Fouriernya adalah :

$$X(\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

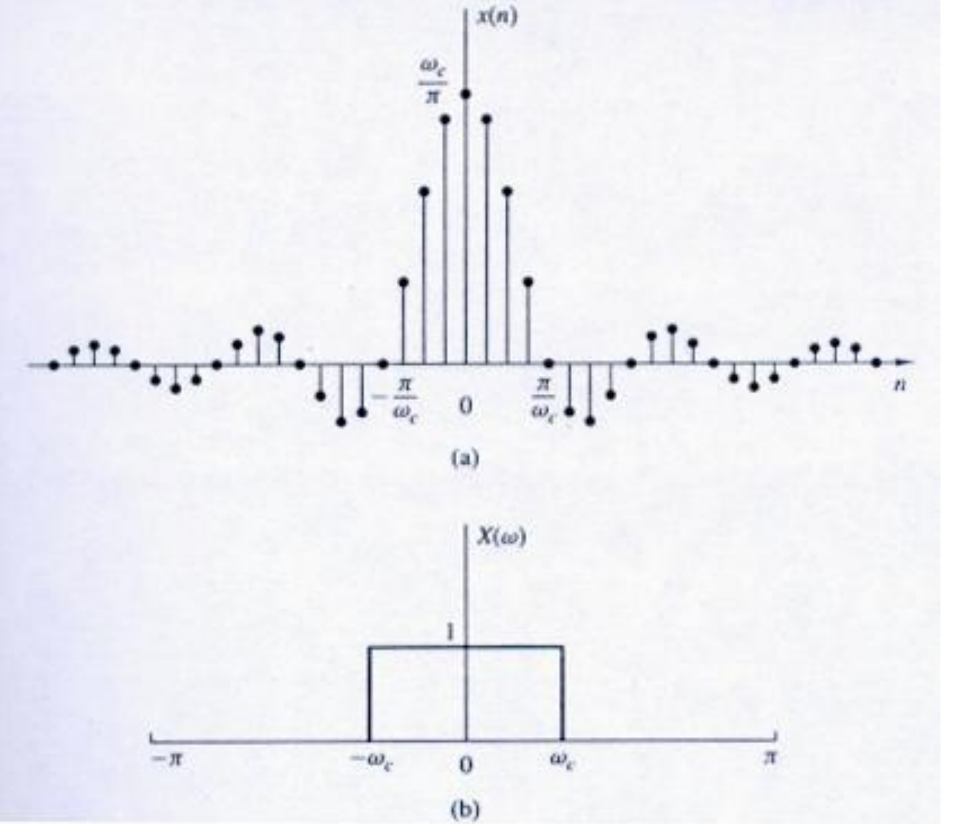
Jawab :

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$

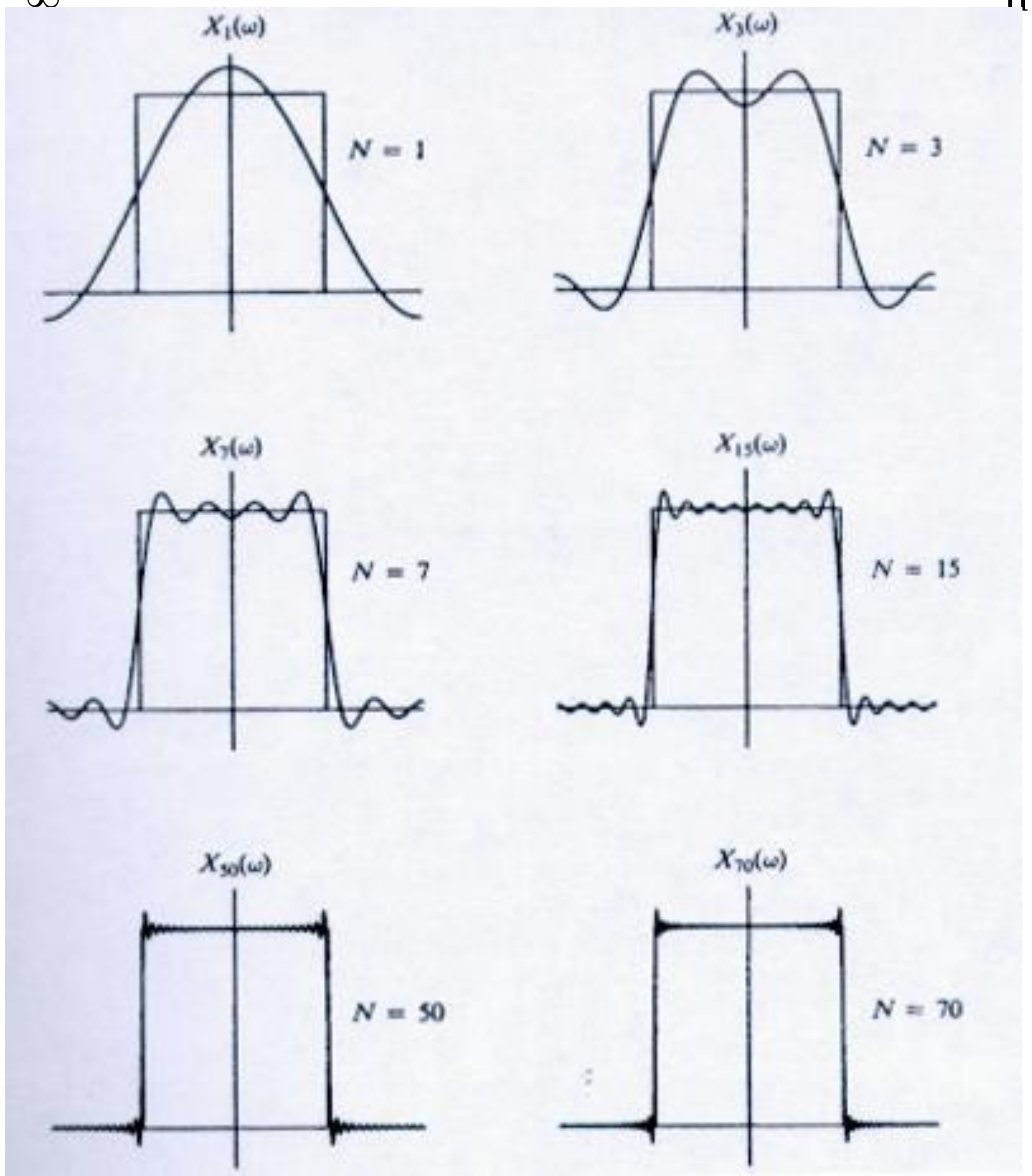
$$n = 0 \quad \rightarrow \quad x(0) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi}$$

$$n \neq 0 \quad \rightarrow \quad x(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{1}{2\pi}\frac{1}{jn} e^{j\omega n}\Big|_{-\omega_c}^{\omega_c}$$

$$x(n) = \frac{1}{\pi n}\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j} = \frac{\sin \omega_c n}{\pi n} = \frac{\omega_c}{\pi}\frac{\sin \omega_c n}{\omega_c n}$$



$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \longrightarrow X_N(\omega) = \sum_{n=-N}^{N} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$



Contoh Soal 7.8

Tentukan transformasi Fourier dari sinyal diskrit :

$$x(n) = \begin{cases} A, & 0 \le n \le L-1 \\ 0, & n \text{ lainnya} \end{cases}$$

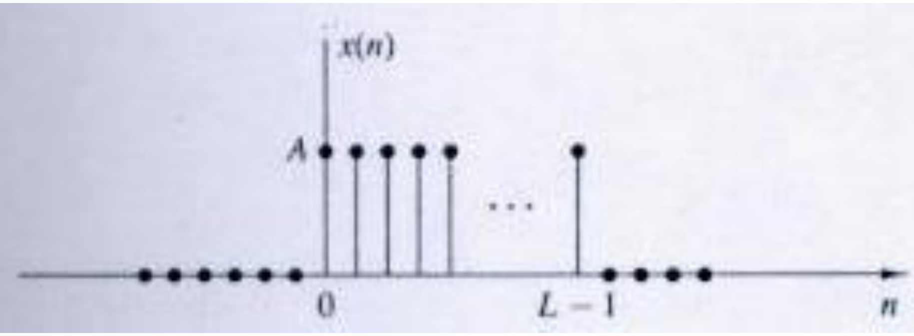


Figure 4.16 Discrete-time rectangular pulse.

Jawab :

$$X(\omega) = \sum_{n=0}^{L-1} A e^{-j\omega n} = A \frac{1 - e^{-j\omega L}}{1 - e^{-j\omega}}$$

$$= A e^{-j(\omega/2)(L-1)} \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

$$X(\omega) = Ae^{-j(\omega/2)(L-1)} \frac{\sin(\omega L/2)}{\sin(\omega/2)} = |X(\omega)| e^{j\Theta(\omega)}$$

$$|X(\omega)| = A\frac{\sin(\omega L/2)}{\sin(\omega/2)}$$ → Respon magnitude

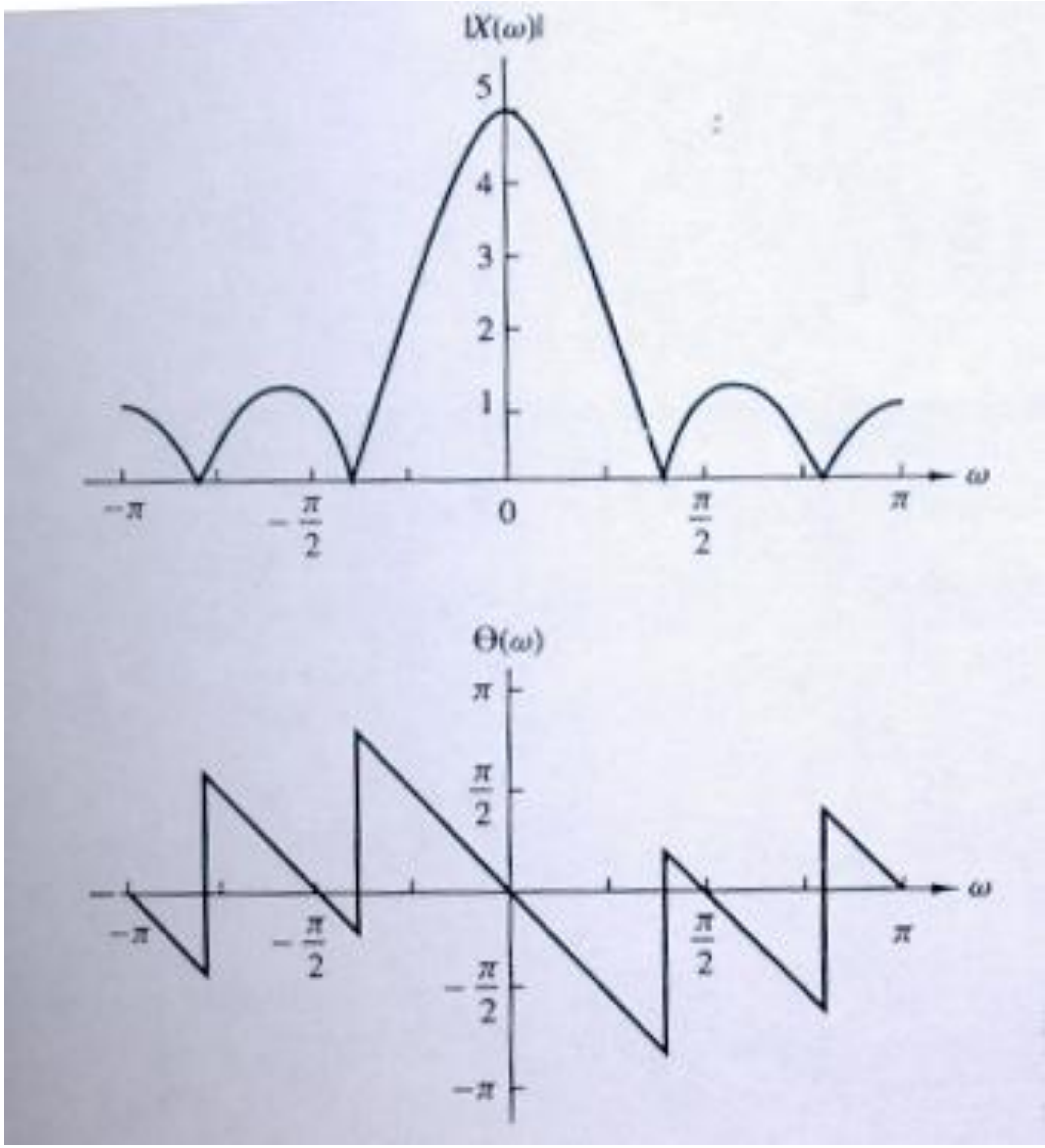
$$\Theta(\omega) = \angle X(\omega) = -\frac{\omega}{2}(L-1)$$ → Respon fasa

A = 1

L = 5

Spektrum magnituda

Spektrum fasa



Hubungan transformasi Z dengan transformasi Fourier

Transformasi Fourier :

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)e^{-z} = \sum_{n=-\infty}^{\infty} x(n)(re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} [x(n)r^{-n}]e^{-j\omega n}$$

$$z = re^{j\omega} \qquad r = |z| \qquad \omega = \angle z$$

$$|z| = 1 \rightarrow \quad r = 1 \qquad \rightarrow \qquad X(z) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = X(\omega)$$

$$\Downarrow$$

Transformasi Fourier pada lingkaran satu = Transformasi Z

Contoh Soal 7.9

Tentukan transformasi Fourier dari : $x(n) = (-1)u(n)$

Jawab :

$$X(z) = \frac{1}{1+z^{-1}} = \frac{z}{z+1}$$

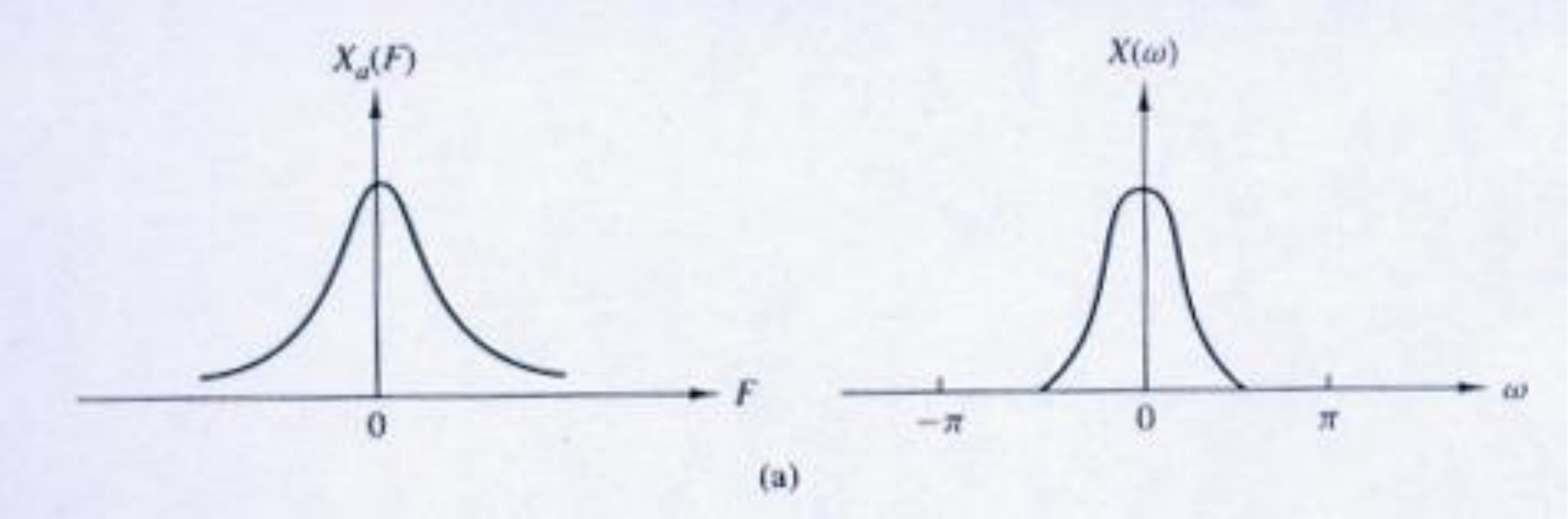
$$X(\omega) = \frac{1}{1+z^{-1}} = \frac{z}{z+1} = \frac{re^{j\omega}}{re^{j\omega}+1}$$

$$= \frac{(e^{j\omega/2})(e^{j\omega/2})}{(e^{j\omega/2})(e^{j\omega/2}+e^{-j\omega/2})}$$

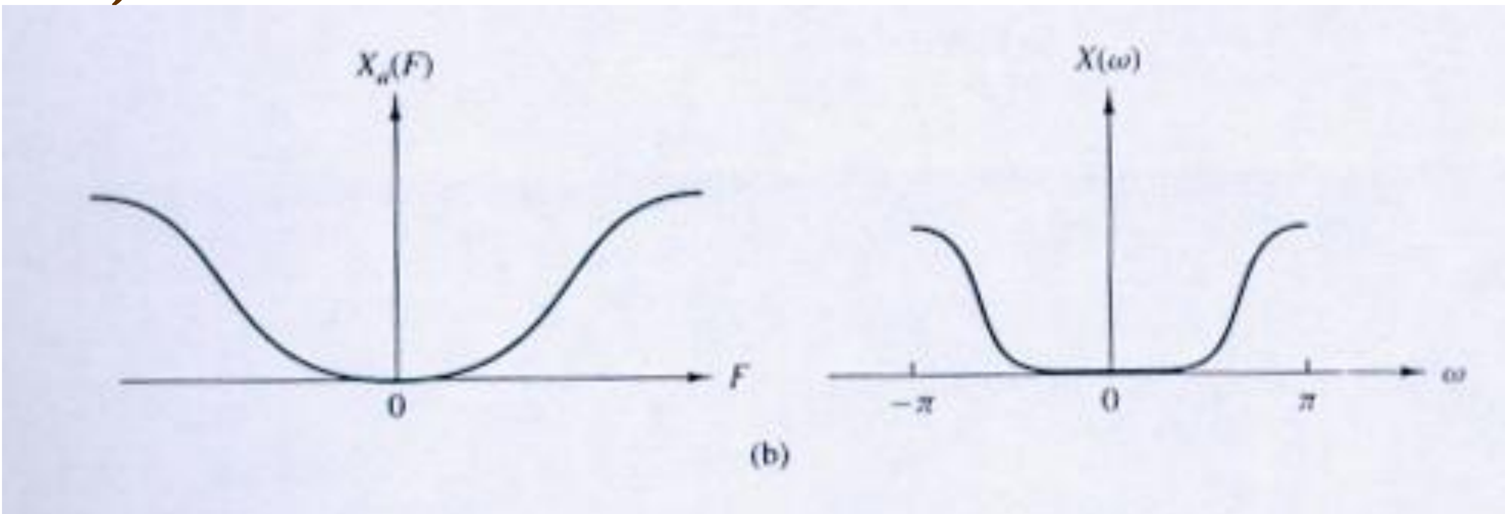
$$= \frac{e^{j\omega/2}}{2\cos(\omega/2)} \qquad \omega \neq 2\pi(k+1/2)$$

Klasifikasi sinyal dalam domain frekuensi

Sinyal frekuensi rendah (Low Pass):

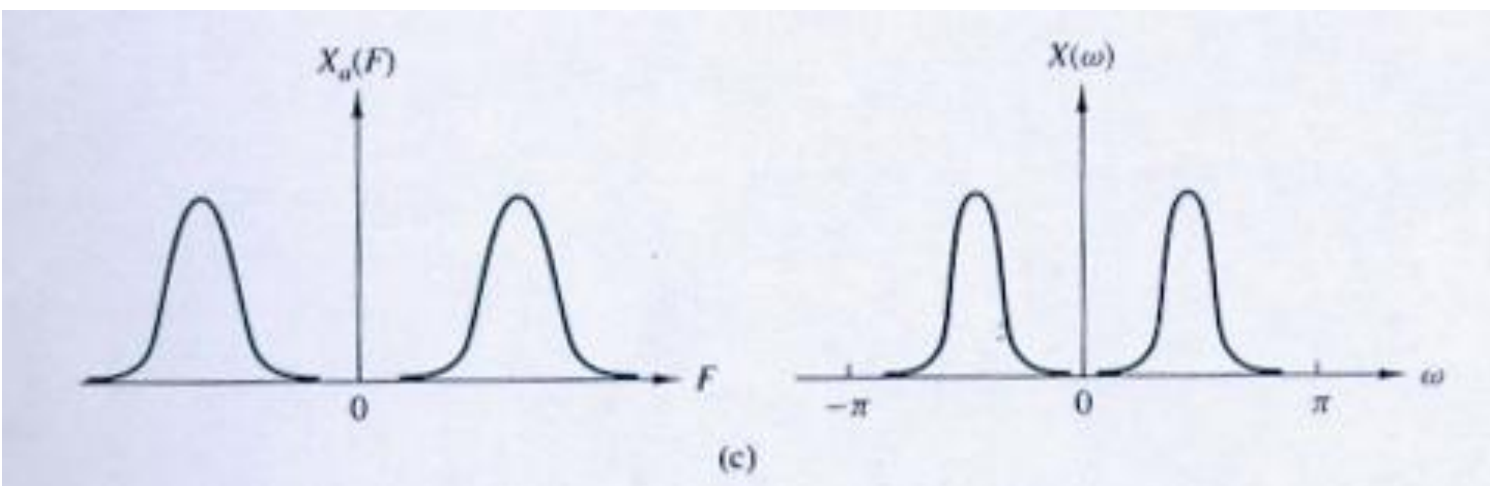


Sinyal frekuensi tinggi (High Pass) :



(b)

Sinyal frekuensi menengah (bandpass signal) :



(c)

Daerah frekuensi pada beberapa sinyal asli

Sinyal-sinyal biologi :

Tipe sinyal	**Daerah frekuensi (Hz)**
Electroretinogram	0 - 20
Electronystagmogram	0 - 20
Pneumogram	0 - 40
Electrocardiogram (ECG)	0 - 100
Electroencephalogram (EEG)	0 - 100
Electromyogram	10 - 200
Aphygmomanogram	0 - 200
Speech	100 - 4000

Sinyal-sinyal seismik :

Tipe sinyal	**Daerah frekuensi (Hz)**
Wind noise	100 - 1000
Seismic exploration signals	10 - 100
Earthquake and nuclear explosion signsld	0.01 - 10
Seismic noise	0,1 - 1

Sinyal-sinyal elektromagnetik :

Tipe sinyal	**Daerah frekuensi (Hz)**
Radio broadcast	$3x10^4 – 3x10^6$
Shortwave radio signals	$3x10^6 – 3x10^{10}$
Radar, sattellite comunications	$3x10^8 – 3x10^{10}$
Infrared	$3x10^{11} – 3x10^{14}$
Visible light	$3,7x10^{14} – 7,7x10^{14}$
Ultraviolet	$3x10^{15} – 3x10^{16}$
Gamma rays and x-rays	$3x10^{17} – 3x10^{18}$

Sifat-sifat transformasi Fourier

- **Linieritas**
- **Pergeseran waktu**
- **Pembalikan waktu**
- **Teorema konvolusi**
- **Pergeseran frekuensi**
- **Diferensiasi frekuensi**

Linieritas

$$F\{x_1(n)\} = X_1(\omega) \qquad F\{x_2(n)\} = X_2(\omega)$$

$$x(n) = a_1 x_1(n) + a_2 x_2(n)$$

$$F\{x(n)\} = X(\omega) = a_1 X_1(\omega) + a_2 X_2(\omega)$$

Contoh Soal 7.11

Tentukan transformasi Fourier dari : $x(n) = a^{|n|} \qquad -1 < a < 1$

Jawab :

$$x(n) = x_1(n) + x_2(n)$$

$$x_1(n) = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases} \qquad x_2(n) = \begin{cases} a^{-n}, & n < 0 \\ 0, & n \geq 0 \end{cases}$$

$$X_1(\omega) = \sum_{n=-\infty}^{\infty} x_1(n)e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

$$= \frac{1}{1 - ae^{-j\omega}}$$

$$X_2(\omega) = \sum_{n=-\infty}^{\infty} x_2(n)e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} = \sum_{n=-\infty}^{-1} (ae^{j\omega})^{-n}$$

$$= \sum_{k=1}^{\infty} (ae^{j\omega})^k = \frac{ae^{j\omega}}{1 - ae^{j\omega}}$$

$$X(\omega) = X_1(\omega) + X_2(\omega) = \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}}$$

$$= \frac{1 - ae^{j\omega} + ae^{j\omega} - a^2}{1 - (ae^{j\omega} + ae^{-j\omega}) + a^2} = \frac{1 - a^2}{1 - 2a\cos\omega + a^2}$$

Pergeseran waktu

$$F\{x_1(n)\} = X_1(\omega)$$

$$x(n) = x_1(n-k) \quad \rightarrow \quad F\{x(n)\} = e^{-j\omega k} X_1(\omega)$$

Pembalikan waktu

$$F\{x_1(n)\} = X_1(\omega)$$

$$x(n) = x_1(-n) \quad \rightarrow \quad F\{x(n)\} = X_1(-\omega)$$

Teorema konvolusi

$$F\{x_1(n)\} = X_1(\omega) \qquad F\{x_2(n)\} = X_2(\omega)$$

$$x(n) = x_1(n) * x_1(n) \quad \rightarrow \quad F\{x(n)\} = X_1(\omega)X_2(\omega)$$

Contoh Soal 7.12

Tentukan konvolusi antara $x_1(n)$ dan $x_2(n)$, dengan :

$$x_1(n) = x_2(n) = \{1, \underset{\uparrow}{1}, 1\}$$

Jawab :

$$X_1(\omega) = \sum_{n=-\infty}^{\infty} x_1(n)e^{-j\omega n} = \sum_{n=-1}^{1} e^{-j\omega n}$$

$$= 1 + e^{-j\omega} + e^{-j\omega} = 1 + 2\cos\omega$$

$$X_1(\omega) = X_2(\omega) = 1 + 2\cos\omega$$

$$X(\omega) = X_1(\omega)X_2(\omega) = (1 + 2\cos\omega)^2$$

$$= 1 + 4\cos\omega + 4\cos^2\omega$$

$$= 1 + 4\cos\omega + 4\left(\frac{1 + \cos 2\omega}{2}\right)$$

$$= 3 + 4\cos\omega + 2\cos 2\omega$$

$$= 3 + 2(e^{j\omega} + e^{-j\omega}) + (e^{j2\omega} + e^{-j2\omega})$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = e^{-j2\omega} + 2e^{-j\omega} + 3 + 2e^{j\omega} + e^{j2\omega}$$

$$x(n) = \{1\ 2\ \underset{\uparrow}{3}\ 2\ 1\}$$

Pergeseran frekuensi

$$F\{x_1(n)\} = X_1(\omega)$$

$$x(n) = e^{j\omega_o n} x_1(n) \quad \rightarrow \quad F\{x(n)\} = X_1(\omega - \omega_o)$$

Diferensiasi frekuensi

$$F\{x_1(n)\} = X_1(\omega) \qquad x(n) = nx_1(n)$$

$$X_1(\omega) = \sum_{n=-\infty}^{\infty} x_1(n)e^{-j\omega n}$$

$$\frac{dX_1(\omega)}{d\omega} = \frac{d}{d\omega}\sum_{n=-\infty}^{\infty} x_1(n)e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x_1(n)\frac{d}{d\omega}e^{-j\omega n}$$

$$= -j\sum_{n=-\infty}^{\infty} nx_1(n)e^{-j\omega n} = -jF\{nx_1(n)\}$$

$$F\{x(n)\} = j\frac{dX_1(\omega)}{d\omega}$$

Domain frekuensi sistem LTI

- **Fungsi respon frekuensi**
- **Respon steady-state**
- **Hubungan antara fungsi sistem dan fungsi respon frekuensi**
- **Komputasi dari fungsi respon frekuensi**

Fungsi respon frekuensi

Eigen function

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

Input kompleks $\rightarrow$ $x(n) = Ae^{j\omega n}$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)Ae^{j\omega(n-k)} = A\sum_{k=-\infty}^{\infty}[h(k)Ae^{-j\omega k}]e^{j\omega n}$$

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \quad \rightarrow \quad y(n) = AH(\omega)e^{j\omega n}$$

Eigen value

Contoh Soal 7.12

Respon impuls dari suatu sistem LTI adalah :

$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

Tentukan outputnya bila mendapat input : $x(n) = Ae^{j\pi n/2}$

Jawab :

$$F\{h(n)\} = H(\omega) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}e^{-j\omega}\right)^n$$

$$H(\omega) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} \quad \rightarrow \quad H(\omega) = \frac{1}{1-\frac{1}{2}e^{-j\pi/2}} = \frac{1}{1+j\frac{1}{2}}$$

$$H(\omega) = \frac{1}{1 + j\frac{1}{2}} = \frac{2}{\sqrt{5}} e^{-j26,6^o}$$

$$y(n) = AH(\omega)e^{j\omega n}$$

$$= A\frac{2}{\sqrt{5}} e^{-j26,6^o} e^{j\pi n/2} = \frac{2A}{\sqrt{5}} e^{(\pi n/2 - 26,6^o)}$$

Amplituda

Fasa

Frekuensi

$$x(n) = Ae^{j\pi n} \quad \rightarrow \quad H(\pi) = \frac{1}{1 - \frac{1}{2}e^{-j\pi}} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}$$

$$y(n) = \frac{2}{3} Ae^{j\pi n}$$

$$H(\omega) = H_R(\omega) + jH_I(\omega)$$

$$= \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k} = \sum_{k=-\infty}^{\infty} h(k)(\cos\omega k - j\sin\omega k)$$

$$H_R(\omega) = \sum_{k=-\infty}^{\infty} h(k)\cos\omega k \quad \rightarrow \quad H_R(-\omega) = H_R(\omega)$$

$$H_I(\omega) = -\sum_{k=-\infty}^{\infty} h(k)\sin\omega k \quad \rightarrow \quad H_I(-\omega) = -H_I(\omega)$$

$$|H(\omega)| = \sqrt{H_R^2(\omega) + H_I^2(\omega)}$$

$$\angle H(\omega) = \Theta(\omega) = tg^{-1}\frac{H_I(\omega)}{H_I(\omega)}$$

$$x_1(n) = Ae^{j\omega n} \quad \rightarrow \quad y_1(n) = A|H(\omega)|e^{j\Theta(\omega)}e^{j\omega n}$$

$$x_2(n) = Ae^{-j\omega n} \quad \rightarrow \quad y_2(n) = A|H(-\omega)|e^{j\Theta(-\omega)}e^{-j\omega n}$$

$$= A|H(\omega)|e^{-j\Theta(\omega)}e^{-j\omega n}$$

$$x(n) = \frac{1}{2}[x_1(n) + x_2(n)] = \frac{1}{2}[Ae^{j\omega n} + Ae^{-j\omega n}] = A\cos\omega n$$

$$y(n) = \frac{1}{2}[y_1(n) + y_2(n)] = A|H(\omega)|\cos[\omega n + \Theta(\omega)]$$

$$x(n) = \frac{1}{j2}[x_1(n) - x_2(n)] = \frac{1}{j2}[Ae^{j\omega n} - Ae^{-j\omega n}] = A\sin\omega n$$

$$y(n) = \frac{1}{j2}[y_1(n) - y_2(n)] = A|H(\omega)|\sin[\omega n + \Theta(\omega)]$$