SIFAT-SIFAT DAN APLIKASI DFT

- Pencuplikan dalam domain frekuensi
- Discrete Fourier Transform (DFT)
- DFT sebagai Transformasi Linier
- Sifat-sifat DFT
- Linear Filtering berdasarkan DFT

Pencuplikan dalam domain frekuensi

x(n) sinyal aperiodik dengan energi terbatas

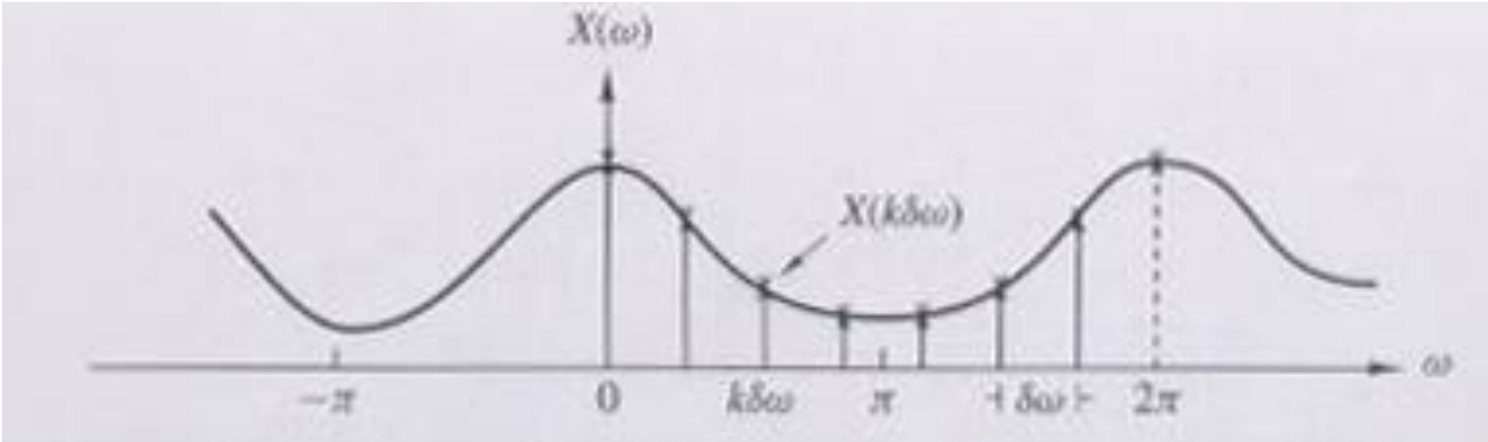
Respon frekuensi : $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

- **Fungsi kontinyu**
- **Periodik dengan perioda 2π**

  $0 \leq \omega < 2\pi \qquad -\pi \leq \omega < \pi$

- **Dicuplik dengan :**

  $$\delta\omega = \frac{2\pi}{N}$$



$$\delta\omega = \frac{2\pi}{N} \quad \Longrightarrow \quad \omega_k = \frac{2\pi}{N}k \qquad k = 0, 1, \cdots, N-1$$

$$X(\omega_k) = X(k\delta\omega) = X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi kn/N}$$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi kn/N} \qquad k = 0, 1, \cdots, N-1$$

$$X\left(\frac{2\pi}{N}k\right) = \cdots + \sum_{n=-N}^{-1} x(n)e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$$

$$+ \sum_{n=N}^{2N-1} x(n)e^{-j2\pi kn/N} + \cdots$$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{m=-\infty}^{\infty} \sum_{n=mN}^{mN+N-1} x(n)e^{-j2\pi kn/N}$$

$$n \to n - mN : X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} \left[\sum_{m=-\infty}^{\infty} x(n-mN) \right] e^{-j2\pi kn/N}$$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi kn/N}$$

$$x_p(n) = \sum_{m=-\infty}^{\infty} x(n - mN)$$

Perulangan periodik dari x(n) setiap N cuplikan

x_p (n) fungsi periodik dengan perioda N

Dapat dinyatakan dalam deret Fourier :

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \qquad n = 0, 1, \cdots, N-1$$

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n) e^{-j2\pi kn/N} \qquad k = 0, 1, \cdots, N-1$$

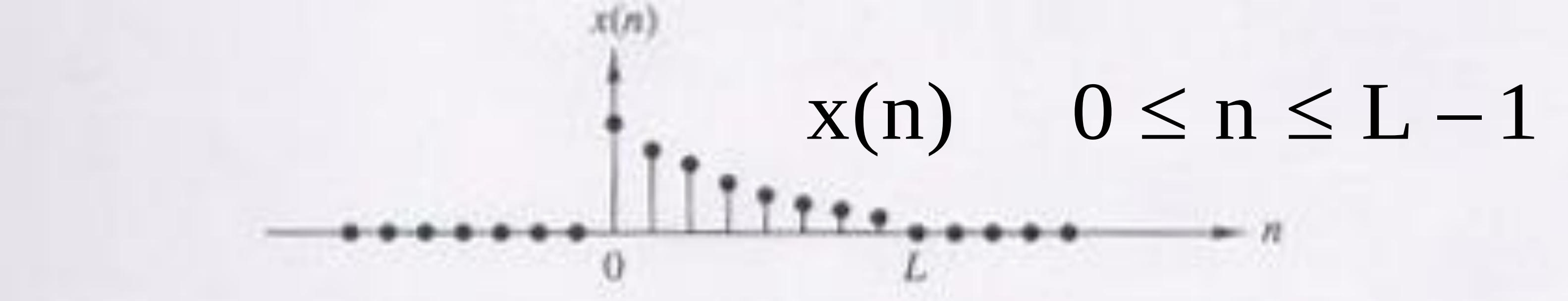
$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n) e^{-j2\pi kn/N}$$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi kn/N}$$

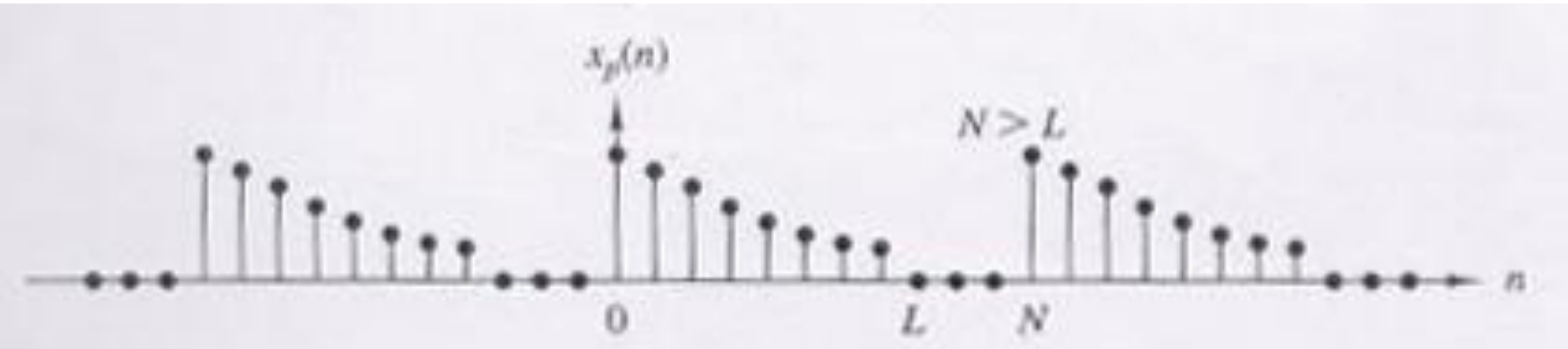
$$c_k = \frac{1}{N} X\left(\frac{2\pi}{N}k\right) \qquad k = 0, 1, \cdots, N-1$$

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \qquad n = 0, 1, \cdots, N-1$$

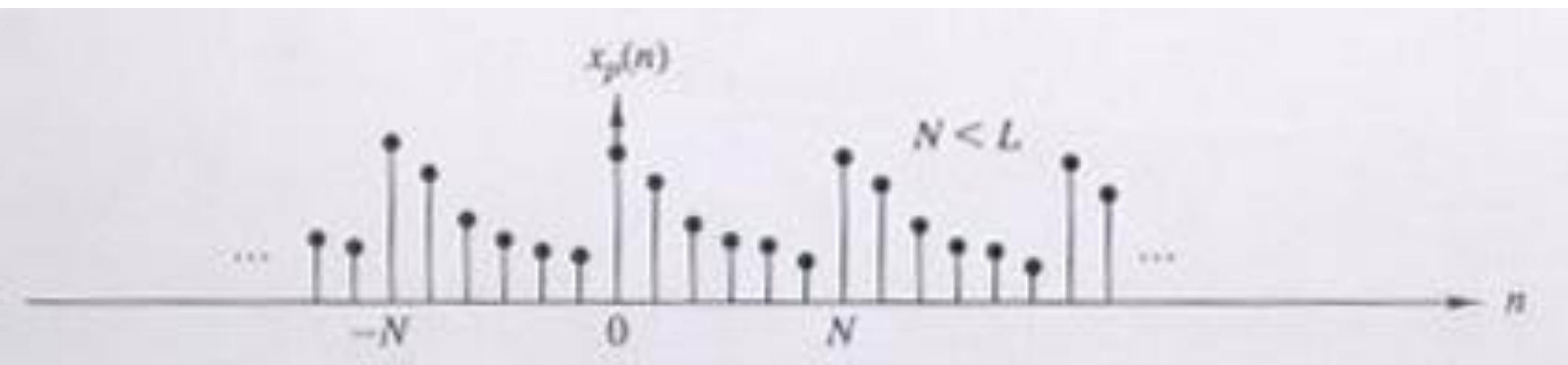
$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N} \qquad n = 0, 1, \cdots, N-1$$



$$x(n) \quad 0 \le n \le L-1$$



$$N \ge L$$



$$N < L$$

$$x(n) = \begin{cases} x_p(n) & 0 \le n \le N-1 \\ 0 & n \text{ lainnya} \end{cases}$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right)e^{j2\pi kn/N} \qquad n = 0, 1, \cdots, N-1$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$X(\omega) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right)e^{j2\pi kn/N}\right]e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} X\left(\frac{2\pi}{N}k\right)\left[\frac{1}{N}\sum_{k=0}^{N-1} e^{-j(\omega-2\pi k/N)n}\right]$$

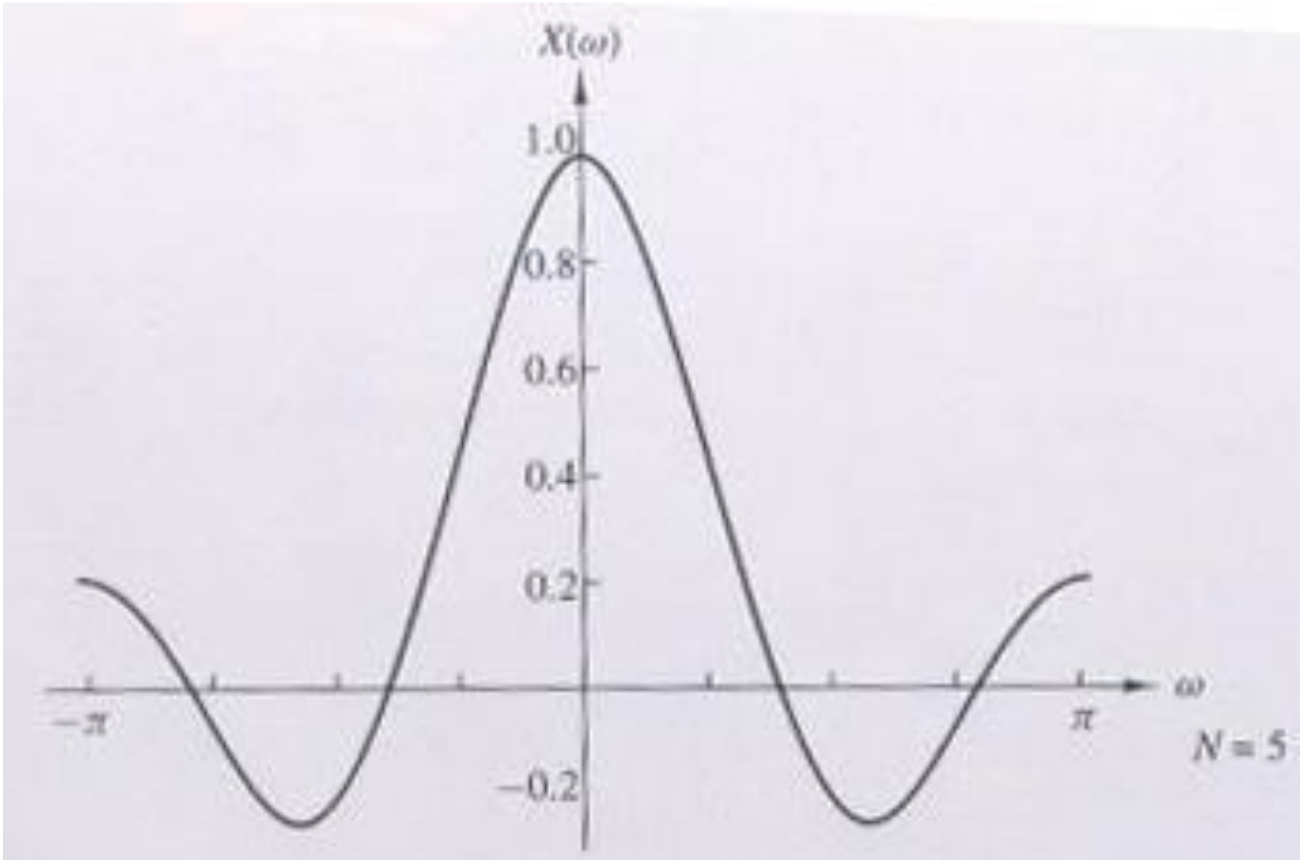
$$P(\omega) = \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1}{N}\frac{1-e^{-j\omega N}}{1-e^{-j\omega}} \qquad \Rightarrow \quad P\left(\omega - \frac{2\pi}{N}k\right)$$

$$P(\omega) = \frac{1}{N}\frac{1-e^{-j\omega N}}{1-e^{-j\omega}} = \frac{1}{N}\frac{e^{-j\omega N/2}(e^{j\omega N/2} - e^{-j\omega N/2})}{e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2})}$$

$$P(\omega) = \frac{\sin(\omega N/2)}{N\sin(\omega/2)}e^{-j\omega(N-1)/2}$$

$$X(\omega) = \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right)P\left(\omega - \frac{2\pi}{N}k\right)$$

$$P\left(\frac{2\pi}{N}k\right) = \begin{cases} 1 & k = 0 \\ 0 & k = 1, 2, \cdots, N-1 \end{cases}$$



$$\frac{\sin(\omega N / 2)}{\sin(\omega / 2)}$$

Contoh Soal 10.1

Diketahui sinyal diskrit : $x(n) = a^n u(n) \qquad 0 < a < 1$

Spektrum sinyal ini dicuplik pada frekuensi-frekuensi :

$$\omega_k = \frac{2\pi k}{N} \qquad k = 0, 1, \cdots, N-1$$

Tentukan hasil rekonstruksi spektrumnya untuk a = 0,8 pada N = 5 dan N = 50

Jawab :

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \frac{1}{1 - ae^{-j\omega}}$$

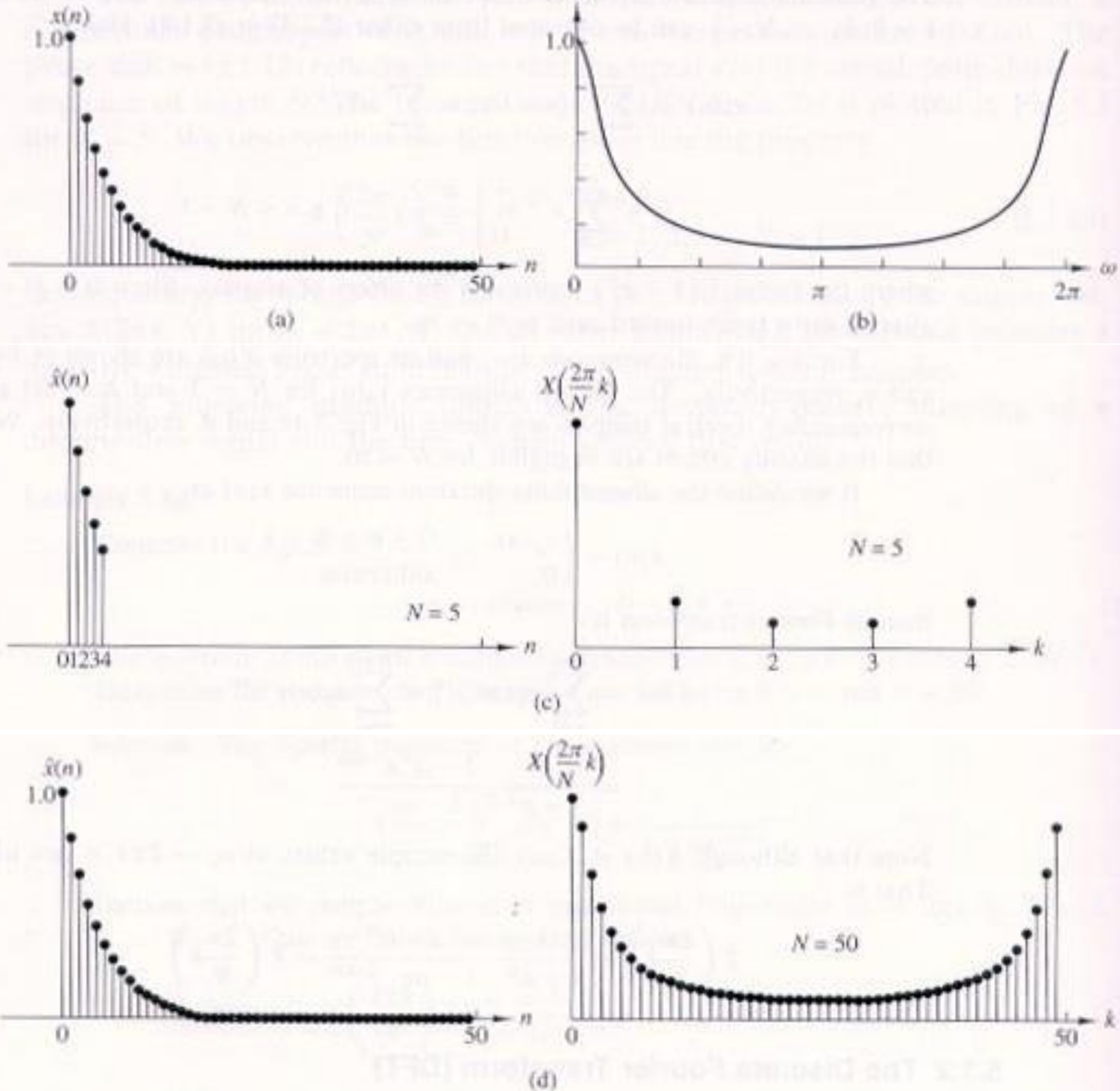
$$X(\omega) = \frac{1}{1 - ae^{-j\omega}}$$

$$X(\omega_k) = X\left(\frac{2\pi k}{N}\right) = \frac{1}{1 - ae^{-j2\pi k/N}}$$

$$x_p(n) = \sum_{m=-\infty}^{\infty} x(n - mN) = \sum_{m=-\infty}^{0} a^{n-mN}$$

$$= a^n \sum_{m=-\infty}^{0} a^{-mN} = a^n \sum_{m=0}^{\infty} a^{mN} = a^n \frac{1}{1 - a^N}$$

Efek aliasing → 0 bila N → ∞



x(n)
1.0
0
50
n
(a)
|X(ω)|
1.0
0
π
2π
ω
(b)
x̃(n)
N = 5
01234
n
X(2π/N k)
N = 5
0
1
2
3
4
k
(c)
x̃(n)
1.0
0
50
n
X(2π/N k)
N = 50
0
50
k
(d)

❑ Discrete Fourier Transform (DFT)

$$x(n) \qquad 0 \le n < L-1$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \qquad 0 \le \omega < 2\pi$$

$$X(\omega) = \sum_{n=0}^{L-1} x(n)e^{-j\omega n}$$

$$X\left(\frac{2\pi}{N}k\right) = X(k) = \sum_{n=0}^{L-1} x(n)e^{-j2\pi kn/N} \qquad N \ge L$$

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \qquad k = 0, 1, \cdots, N-1$$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \qquad k = 0, 1, \cdots, N-1$$

Discrete Fourier Transform (DFT)

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{-j2\pi kn/N} \qquad n = 0, 1, \cdots, N-1$$

Inverse Discrete Fourier Transform (IDFT)

Contoh Soal 10.2

Diketahui deret diskrit x(n) dengan panjang terbatas L :

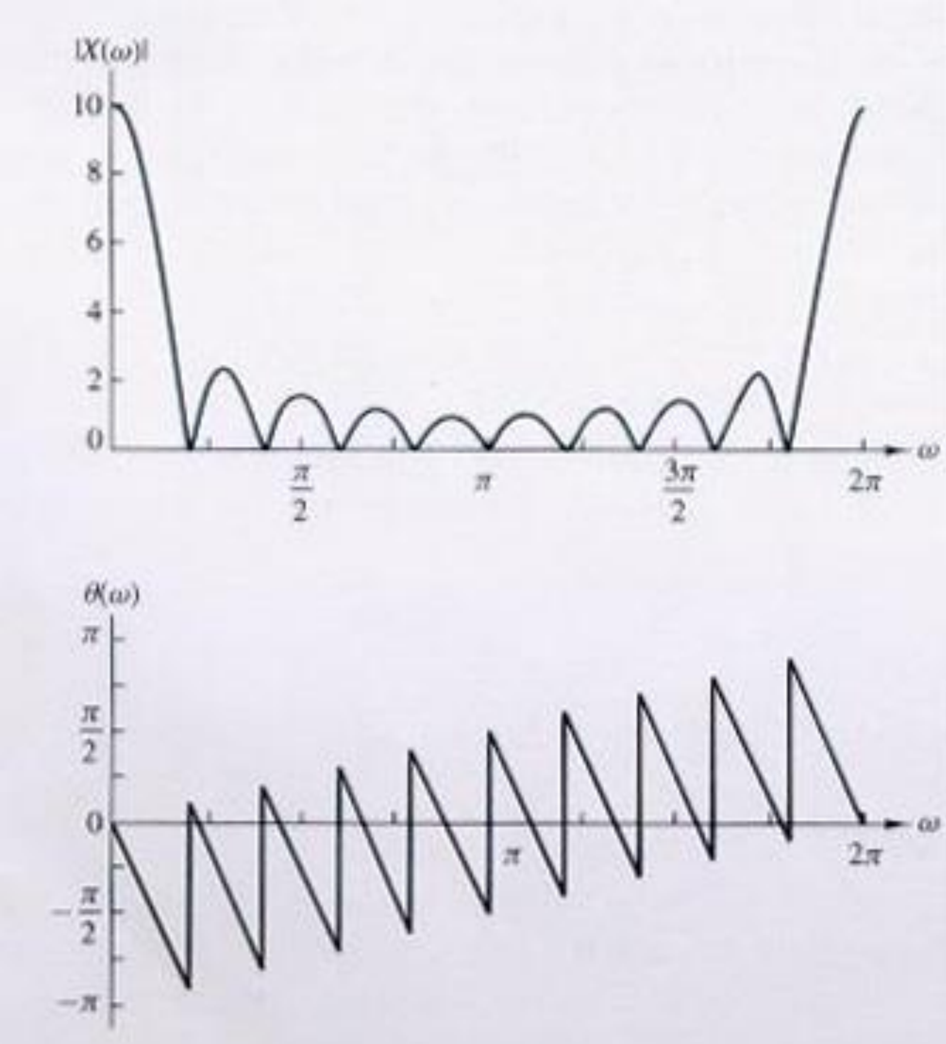
$$x(n) = \begin{cases} 1 & 0 \le n < L-1 \\ 0 & \text{lainnya} \end{cases}$$

Tentukan N-point DFT dari x(n) untuk $N \ge L$

Jawab :

$$X(\omega) = \sum_{n=0}^{L-1} x(n) e^{-j\omega n} = \sum_{n=0}^{L-1} e^{-j\omega n}$$

$$= \frac{1 - e^{-j\omega L}}{1 - e^{-j\omega}} = e^{-j\omega(L-1)/2} \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$



$$X(\omega) = e^{-j\omega(L-1)/2} \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

N-point DFT :

$$X(k) = \frac{1 - e^{-j2\pi kL/N}}{1 - e^{j2\pi k/N}} \qquad k = 0,1,\cdots N-1$$

$$= \frac{\sin(\pi kL/N)}{\sin(\pi k/N)} e^{-j\pi k(L-1)/N}$$

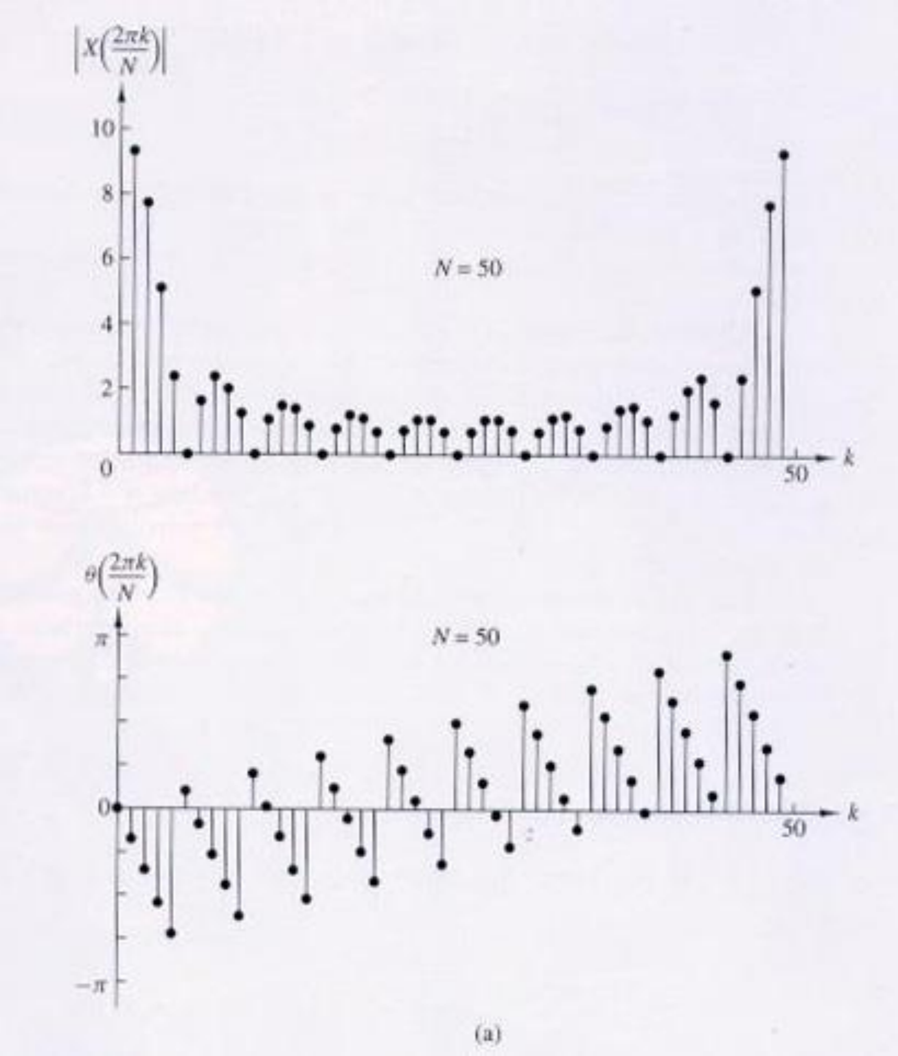
Bila N = L DFT menjadi :

$$X(k) = L, \qquad k = 0$$

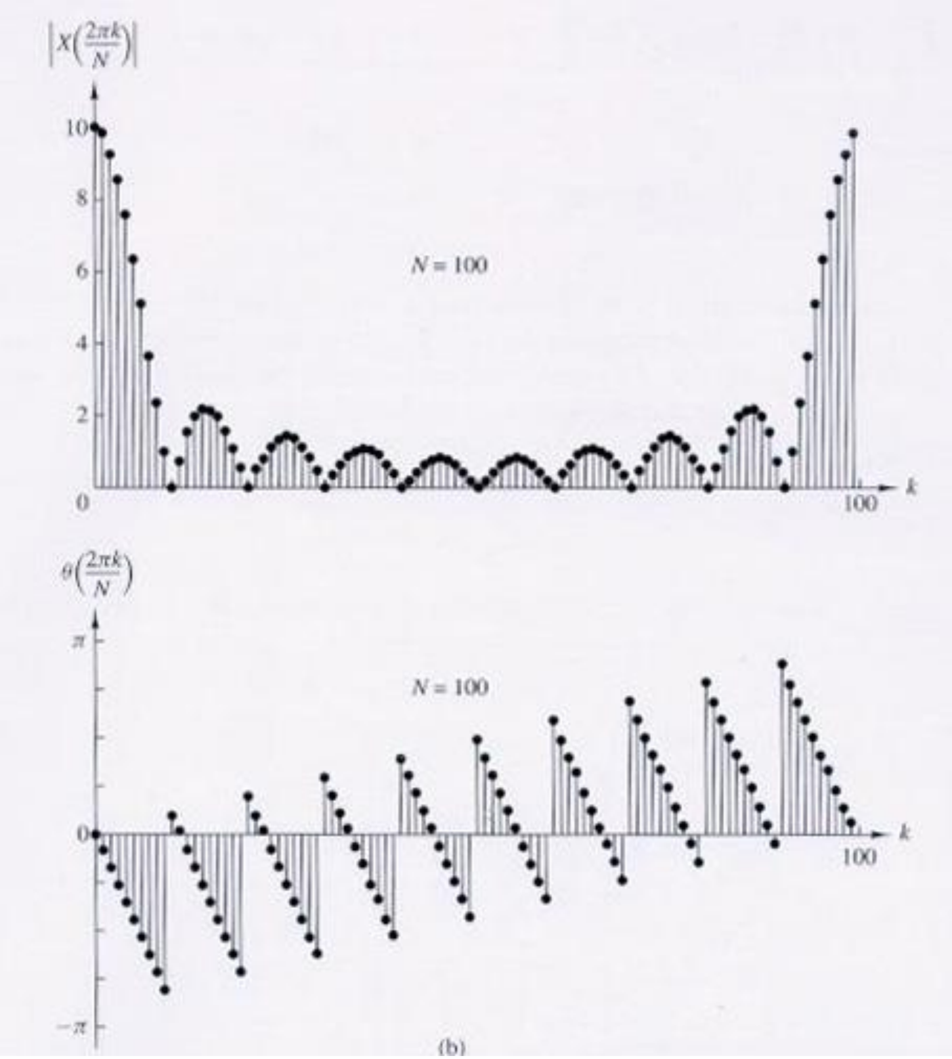
$$= 0, \qquad k = 1,2,\cdots L-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{-j2\pi kn/N} \qquad n = 0,1,\cdots, N-1$$

$$x(n) = \frac{1}{10} 10 = 1$$



$L = 10 \quad N = 50$



$L = 10 \quad N = 100$

❑DFT dan IDFT sebagai transformasi linier

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad k = 0, 1, \cdots, N-1$$

$$W_N = e^{-j2\pi/N}$$

DFT sebagai transformasi linier dari x(n)

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{-j2\pi kn/N} = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} = \qquad n = 0, 1, \cdots, N-1$$

IDFT sebagai transformasi linier dari X(k)

$$x_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \qquad X_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$W_N = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & \cdots & W_4^0 \\ W_4^0 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ W_4^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ W_4^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

$$X_N = W_N x_N$$

$$X_N = W_N x_N$$

$$W_N^{-1} X_N = W_N^{-1} W_N x_N \quad \rightarrow \quad x_N = W_N^{-1} X_N$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}$$

$$W_N^{-kn} = \left(W_N^{kn}\right)^*$$

$$x_N = \frac{1}{N} W_N^* X_N \quad \rightarrow \quad W_N^{-1} = \frac{1}{N} W_N^*$$

Contoh Soal 10.3

Diketahui deret diskrit x(n) dengan panjang terbatas 4 :

$$x(n) = (0 \quad 1 \quad 2 \quad 3)$$

Tentukan 4-point DFT dari x(n)

Jawab :

$$W_N = e^{-j2\pi/N} \longrightarrow W_N^k = e^{-j2\pi k/N}$$

$$W_N^{k+N/2} = e^{-j2\pi(k+N/2)/N} = e^{-j\pi}e^{-j2\pi k/N} = -W_N^k$$

$$x_N = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad W_N = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}$$

$$W_N^{k+N/2} = e^{-j2\pi(k+N/2)/N} = e^{-j\pi}e^{-j2\pi k/N} = -W_N^k$$

$$W_4^0 = e^0 = 1$$

$$W_4^1 = e^{-j2\pi/4} = e^{-j\pi/2} = -j$$

$$W_4^2 = e^{-j2\pi 2/4} = e^{-j\pi} = -1 \qquad W_4^2 = W_4^{0+4/2} = -W_4^0 = -1$$

$$W_4^3 = e^{-j2\pi 3/4} = e^{-j3\pi/2} = j \qquad W_4^3 = W^{1+4/2} = -W_4^1 = j$$

$$W_4^4 = W^{2+4/2} = -W_4^2 = -W_4^{0+4/2} = -(-W_4^0) = W_4^0 = 1$$

$$W_4^6 = W_4^2 = -1$$

$$W_4^9 = W_4^1 = -j$$

$$W_N = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$X_4 = W_N x_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$x_4 = \frac{1}{N} W_N^* X_4 = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$